Exam. Code : 211003
Subject Code : 5498

M.Sc. Mathematics 3rd Semester

MATH-572 : TOPOLOGY–I

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **two** questions from each Unit. All questions carry **10** marks each.

UNIT–I

1. Prove that Int (A) = $C\left(\overline{C(A)}\right)$ for any set A, where C(A) denotes the complement of A in X. Further prove that A is open if and only if A = Int (A).
2. Prove that every separable metric space is 2^{nd} countable.
3. Let X and Y be topological spaces and $f : X \to Y$ a map. Prove that if $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every $B \subset Y$, then inverse image of each closed set in Y is closed in X.
4. If Y is a space satisfying the second axiom of countability, then prove that every open covering $\{U_\alpha\}$ has a countable subcovering.

UNIT–II

5. (i) Define a connected space and prove that continuous image of a connected space is connected.

   (ii) Prove that a topological space is locally connected if the components of every open subspace of X are open in X.

6. Prove that a topological space is disconnected if and only if there exist a continuous map of X onto the discrete two point space [0, 1].

7. If X is an arbitrary topological space then prove that :
   (i) Each point in X is contained in exactly one component of X.
   (ii) Each connected subspace of X is contained in a component of X.
   (iii) A connected subspace of X which is both open and closed is a component of X.

8. Let $(X, \Im)$ be a space and $(Y, \Im_Y)$ a subspace. Then prove that :

   $\overline{A_Y} = Y \cap \overline{A}$; $A'_Y = Y \cap A'$; $Y \cap Int(A) \subset Int_Y (A)$; $Fr_Y(A) \subset Y \cap Fr(A)$.

UNIT–III

9. Let $f : X \to Y$ be closed. Then prove that for all $S \subset X$, $\forall$ open U such that $f^{-1}(S) \subset U$ there exist open V in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

10. Let $f : X \to Y$ be a homeomorphism. Prove that for any $A \subset X$ the map $g = f|_A : A \to f(A)$ is also a homeomorphism.

11. (i) If f is a continuous mapping of the topological space X into the topological space Y, and $\{x_n\}$ is a sequence of points of X which converges to the point $x \in X$, then the sequence $\{f(x_n)\}$ converges to the point f(x) in Y.
    (ii) Prove that a map $f : X \to Y$ is open if and only if $\forall A \subset X$, $f(A^o) \subset (f(A))^o$.

12. Let $X = A \cup B$, where A, B are both open or both closed in X. Let $f : X \to Y$ be such that $f|_A$ and $f|_B$ are continuous, then prove that f is also continuous.

UNIT–IV

13. In the product space $\Pi_\alpha Y_\alpha$,
    (i) If S_α is a sub-basis of $(Y_\alpha, \mathfrak{I}_\alpha)$, then prove that the collection $\{<V_\beta> : V_\beta \in S_\beta, \beta \in \Lambda\}$ is a sub-basis for $\Pi_\alpha Y_\alpha$.
    (ii) Let $A_\alpha \subset Y_\alpha$, then A_α has subspace topology on it. Let $\Pi_\alpha A_\alpha$ has product topology $\mathfrak{I}^*$. Considering $\Pi_\alpha A_\alpha$ as a subspace of $\Pi_\alpha Y_\alpha$, $\Pi_\alpha A_\alpha$ has the topology $\mathcal{T}_*$ on it, prove that $\mathfrak{I}^* = \mathcal{T}_*$.

14. (i) Show that infinite product of non trivial discrete spaces is never discrete.
    (ii) Prove that $(\Pi_\alpha A_\alpha)^C = \cup_\alpha (A_\alpha)^C$.

15. Prove that the projection maps are open but they need not be closed.

16. Define the quotient space. Prove that if Y is a quotient space of X and Z is a quotient space of Y then Z is homeomorphic to a quotient space of X.

UNIT–V

17. Show that a closed subspace of a normal space is normal.

18. State and prove Tietz extention theorem.

19. Show that a one to one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.

20. Prove that $\Pi_\alpha\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ is regular if and only if each Y_α is regular.